

# Topological Theories from Virasoro Constraints on the KP Hierarchy

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## ABSTRACT

A conformal field theory can be recovered, via the Kontsevich-Miwa transform, as a solution to the Virasoro constraints on the KP  $\tau$  function. That theory, which we call KM CFT, consists of  $d \leq 1$  matter plus a scalar and a dressing prescription:  $\Delta = 0$  for every primary field. By adding a spin-1  $bc$  system the KM CFT provides a realization of the  $N = 2$  twisted topological algebra. The other twist of the corresponding untwisted  $N = 2$  superconformal theory is a DDK realization of the  $N = 2$  twisted topological algebra.

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# 1 Introduction

To start let me review a couple of concepts about null states. In general, a null state is a state with zero norm. In conformal field theories (CFT's) null states are those states that are both primary and secondary. That is, they are descendants built on primary states by acting with the Virasoro generators (and possibly other generators as well) and they also satisfy the highest weight conditions that define primary states [1].

In CFT's null states are orthogonal to any primary and any secondary state (in the Hilbert space there may also exist states that are neither primary nor secondary and null states need not be orthogonal to them). Therefore when a null state enters a correlator of primary and/or secondary fields, the correlator must vanish.

By using the commutation relation

$$[L_n, \Psi(z)] = \left( z^{n+1} \frac{\partial}{\partial z} + (n+1) \Delta z^n \right) \Psi(z) \quad (1.1)$$

between the Virasoro generators and the primary fields (and other analogous relations if there are other generators), the vanishing of the correlator translates into a differential equation of the same order as the level of the corresponding null vector. We will refer to those differential operators as *decoupling operators* since they express the decoupling of the null states

$$\left( \text{decoupling operator} \right) \left\langle \Psi(z_i) \prod_{j \neq i} \Psi_j(z_j) \right\rangle = 0 . \quad (1.2)$$

Here  $\Psi(z_i)$  is singled out as the primary field on which the null descendant was built.

## 2 CFT's from Virasoro Constraints on the KP Hierarchy

The Virasoro constraints on the KP  $\tau$  function  $L_p \tau = 0$ ,  $p \geq -1$ , are constraints that are compatible with the KP flows and satisfy the Virasoro algebra (half of it). They are given by

$$\begin{aligned} L_{p>0} &= \frac{1}{2} \sum_{s=1}^{p-1} \frac{\partial^2}{\partial t_{p-s} \partial t_s} + \sum_{s \geq 1} s t_s \frac{\partial}{\partial t_{p+s}} + \left( J - \frac{1}{2} \right) (p+1) \frac{\partial}{\partial t_p} \\ L_0 &= \sum_{s \geq 1} s t_s \frac{\partial}{\partial t_s} \\ L_{-1} &= \sum_{s \geq 1} (s+1) t_{s+1} \frac{\partial}{\partial t_s} \end{aligned} \quad (2.1)$$

Where  $J$  is the only degree of freedom. They were written down independently by A. Semikhatov [2] and P. Grinevich and Yu. Orlov [3], although the final form with the degree of freedom  $J$  is due to the first author.

Let us define the Kontsevich-Miwa (KM) transformation as

$$t_r = \frac{1}{r} \sum_j n_j z_j^{-r}, \quad r \geq 1 \quad (2.2)$$

where  $n_j$  are real numbers and the coordinates  $z_j$  belong to a set  $\{z_j\}$  of points on the spectral complex plane of the hierarchy. The KM transformation looks formally like the Miwa transformation but it is not quite the same, since for Miwa the parameters  $n_j$  were essentially integer numbers [4] [5]. He was interested in expressing the bilinear Hirota equations as finite-difference equations in the  $n_j$ . His attention was on these parameters rather than on the points  $z_j$ . On the other hand, more recently Kontsevich [6] used the Miwa transformation paying all his attention to the points  $z_j$  (all the  $n_j$  were set equal to a constant). Since for us the parameters  $n_j$  and the points  $z_j$  are equally important, we call (2.2) the Kontsevich-Miwa transformation.

Intriguingly enough, using the KM transformation, a CFT can be recovered as a solution to the Virasoro constraints (2.1). Namely, for any chosen  $(z_i, n_i)$ ,  $z_i \in \{z_j\}$ , and provided  $2J - 1 = \frac{1}{n_i} - 2n_i$ , the object

$$\left( \sum_{p \geq -1} z_i^{-p-2} \mathbf{L}_p \right) \quad (2.3)$$

(sort of half the energy-momentum tensor) KM-transforms into a decoupling operator [10]

$$\left\{ -\frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{1}{z_j - z_i} \left( n_j \frac{\partial}{\partial z_i} - n_i \frac{\partial}{\partial z_j} \right) \right\}, \quad (2.4)$$

corresponding to a level-2 null vector in a CFT of  $d \leq 1$  matter dressed by a scalar with zero background charge, so that  $c = d + 1$ . The Virasoro generators corresponding to this CFT can be written, therefore, as

$$L_n = L_n^{matter} - \frac{1}{2} \sum_{m \in \mathbf{Z}} : I_{n-m} I_m : \quad (2.5)$$

The eigenvalues of  $L_0$  and  $I_0$  give the conformal weights and  $U(1)$  charges of the fields

$$L_0 |\Psi_j\rangle = \Delta_j |\Psi_j\rangle, \quad I_0 |\Psi_j\rangle = n_j |\Psi_j\rangle. \quad (2.6)$$

Observe that the parameter  $n_j$  plays the role of the  $U(1)$  charge of the field  $\Psi_j(z_j)$  sitting at the location  $z_j$ . This theory, which we call KM CFT, has also a built-in

dressing prescription,  $\Delta_j = -\frac{1}{2}Q_M n_j$ , that we call the KM dressing for obvious reasons ( $Q_M = \sqrt{\frac{1-d}{3}}$  is the background charge of the matter).

The solution to the decoupling equation (1.2) corresponding to the decoupling operator (2.4) is therefore a correlator in which  $\Psi(z_i)$  is a (2,1) or (1,2) dressed primary field with  $U(1)$  charge  $n_i$ , and  $\Psi_j(z_j)$  are fields with  $U(1)$  charges  $n_j$ .

Now let us consider level  $l$  decoupling operators in the KM CFT. These KM-transform back into objects of the form [8]

$$\mathcal{O}_l(\mathbf{L}_p) \left( \sum_{p \geq -1} z_i^{-p-2} \mathbf{L}_p \right) \quad (2.7)$$

where  $\mathcal{O}_l$  is a differential operator, provided

$$2J - 1 = \frac{l-1}{n_i} - \frac{2n_i}{l-1} \quad (2.8)$$

For a given set of Virasoro constraints (a given  $J$ ) this tuning between the level  $l$  and  $n_i$  is a fortunate fact that ensures that for a given  $(z_i, n_i)$  there is at most one level at which the KM transformation of the object (2.3) can take place.

One remark we should make is that the above is exactly true when the decoupling operators correspond to null vectors built on  $(1, l)$  or  $(l, 1)$  primary fields. The solution to the decoupling equation (1.2) is therefore a correlator with  $\Psi(z_i)$  a  $(l, 1)$  or  $(1, l)$  dressed primary field.

### 3 KM and DDK Realizations of the Topological N=2 Twisted Algebra.

Let us concentrate on the KM CFT. By twisting the Virasoro generators in the form

$$\hat{L}_n = L_n + \frac{1}{2}Q_M(n+1)I_n \quad (3.1)$$

we get a universal KM dressing:  $\Delta_j = 0$  for every field  $\Psi_j$ . This twisting modifies also the central charge and the scalar background charge:  $c = 2$  and  $Q_s = Q_M$ , while previously  $c = d + 1$  and  $Q_s = 0$ . Now, by adding a spin-1  $bc$ -system ( $c_{gh} = -2$ ) the KM CFT can be embedded into a  $N = 2$  twisted topological theory [7] [8]. The topological twisted  $N = 2$  algebra reads [12] [13]

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{c}{6}(m^2+m)\delta_{m+n,0}, & & m, n \in \mathbf{Z}. \quad (3.2) \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2+m)\delta_{m+n,0},
\end{aligned}$$

$\mathcal{L}_m$  and  $\mathcal{H}_m$  are the bosonic generators corresponding to the energy-momentum tensor and the topological  $U(1)$  current respectively, while  $\mathcal{Q}_m$  and  $\mathcal{G}_m$  are the fermionic generators corresponding to the BRST current and spin-2 fermionic current. The *topological central charge*  $c$  is the true central charge of the  $N=2$  superconformal algebra.

The KM realization of this algebra is

$$\mathcal{L}_m = \hat{L}_m + l_m, \quad l_m = \sum_{n \in \mathbf{Z}} n :b_{m-n}c_n: \quad (3.3)$$

$$\mathcal{H}_m = - \sum_{n \in \mathbf{Z}} :b_{m-n}c_n: + \sqrt{\frac{3-c}{3}} I_m, \quad \mathcal{Q}_m = b_m, \quad (3.4)$$

$$\mathcal{G}_m = 2 \sum_{p \in \mathbf{Z}} c_{m-p} \hat{L}_p + 2\sqrt{\frac{3-c}{3}} \sum_{p \in \mathbf{Z}} (m-p)c_{m-p} I_p + \sum_{p,r \in \mathbf{Z}} (r-p) :b_{m-p-r}c_r c_p: + \frac{c}{3}(m^2+m)c_m, \quad (3.5)$$

and the chiral primary states split as  $|\Phi\rangle = |\Psi\rangle \otimes |0\rangle_{gh}$ , where  $|\Psi\rangle$  is a primary state of the KM CFT.

The null states of the KM CFT can be recovered from the topological theory as follows [7] [11]. One constructs BRST-invariant ( $\mathcal{Q}_0$ -invariant) topological null states of bosonic type  $|\Xi\rangle^{BQ}$ . Then, after the splitting of the topological generators into their KM components, all ghost contributions cancel out and one is left with

$$|\Xi\rangle_{KM}^{BQ} = |\Upsilon\rangle \otimes |0\rangle_{gh}, \quad (3.6)$$

where  $|\Upsilon\rangle$  is a null state of the KM CFT.

Now it comes the question about the relatives of the KM topological theory. Namely the *mother*  $N=2$  untwisted superconformal theory and the *sister*  $N=2$  twisted topological theory corresponding to the other possible twist of the untwisted superconformal theory. After some computations one finds [8] that the *mother*  $N=2$  superconformal theory, satisfying the superconformal algebra [14]

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{\mathfrak{c}}{12}(m^3-m)\delta_{m+n,0}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{\mathfrak{c}}{3}m\delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_r^\pm] &= \left(\frac{m}{2}-r\right)\mathcal{G}_{m+r}^\pm, & [\mathcal{H}_m, \mathcal{G}_r^\pm] &= \pm\mathcal{G}_{m+r}^\pm, \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} \\
\{\mathcal{G}_r^-, \mathcal{G}_s^+\} &= 2\mathcal{L}_{r+s} - (r-s)\mathcal{H}_{r+s} + \frac{\mathfrak{c}}{3}(r^2-\frac{1}{4})\delta_{r+s,0},
\end{aligned} \tag{3.7}$$

is given by

$$\mathcal{L}_m = L_m + \frac{1}{4}(Q_L + Q_M)(m+1)I_m + l_m, \quad l_m = \sum_r \left(r + \frac{m}{2}\right) :b_{m-r}c_r:, \tag{3.8}$$

$$\mathcal{H}_m = -\sum_r :b_{m-r}c_r: - \frac{1}{2}(Q_L - Q_M)I_m, \quad \mathcal{G}_r^- = b_r, \tag{3.9}$$

$$\mathcal{G}_r^+ = 2\sum_n c_{r-n}L_n + \sum_{s,q} (s-q) :b_{r-s-q}c_sc_q: + \sum_n (Q_L n + (Q_M - Q_L)r + \frac{1}{2}Q_M + \frac{1}{2}Q_L)c_{r-n}I_n + \frac{\mathfrak{c}}{3}(r^2 - \frac{1}{4})c_r. \tag{3.10}$$

where  $Q_M$  and  $Q_L$  are expressed in terms of  $\mathfrak{c}$  as

$$Q_M = -\frac{\mathfrak{c}+3}{\sqrt{3(3-\mathfrak{c})}}, \quad Q_L = \frac{\mathfrak{c}-9}{\sqrt{3(3-\mathfrak{c})}}. \tag{3.11}$$

Here  $\mathcal{G}_r^\pm$  are spin-3/2 supersymmetric currents (the ghosts are now a spin-3/2  $bc$  system). The twist that gives rise to the KM topological theory is

$$\begin{aligned}
\mathcal{L}_m^{(1)} &= \mathcal{L}_m + \frac{1}{2}(m+1)\mathcal{H}_m, \\
c_m^{(1)} &= c_{m+\frac{1}{2}}, & b_m^{(1)} &= b_{m-\frac{1}{2}}, \\
\mathcal{H}_m^{(1)} &= \mathcal{H}_m, \\
\mathcal{G}_m^{(1)} &= \mathcal{G}_{m+\frac{1}{2}}^+, & \mathcal{Q}_n^{(1)} &= \mathcal{G}_{n-\frac{1}{2}}^-,
\end{aligned} \tag{3.12}$$

while the other twist

$$\begin{aligned}
\mathcal{L}_m^{(2)} &= \mathcal{L}_m - \frac{1}{2}(m+1)\mathcal{H}_m, \\
c_m^{(2)} &= c_{m-\frac{1}{2}}, & b_m^{(2)} &= b_{m+\frac{1}{2}}, \\
\mathcal{H}_m^{(2)} &= -\mathcal{H}_m, \\
\mathcal{G}_m^{(2)} &= \mathcal{G}_{m+\frac{1}{2}}^-, & \mathcal{Q}_n^{(2)} &= \mathcal{G}_{n-\frac{1}{2}}^+,
\end{aligned} \tag{3.13}$$

provides a DDK realization [9] of the topological algebra. That is, it consists of minimal matter dressed by the Liouville ( $\Delta = 1$  for all the primary fields, and  $Q_s = \sqrt{\frac{25-d}{3}} = Q_L$ )

and a spin-2  $bc$  system ( $c_{gh} = -26$ ). In components, the DDK realization of the  $N = 2$  twisted topological algebra reads [7] [8]

$$\mathcal{L}_m = \hat{L}_m + l_m, \quad l_m \equiv \sum_{n \in \mathbf{Z}} (m+n) :b_{m-n}c_n: \quad (3.14)$$

$$\mathcal{H}_m = \sum_{n \in \mathbf{Z}} :b_{m-n}c_n: - \sqrt{\frac{3-c}{3}} I_m, \quad \mathcal{G}_m = b_m, \quad (3.15)$$

$$\mathcal{Q}_m = 2 \sum_{p \in \mathbf{Z}} c_{m-p} \hat{L}_p + \sum_{p,r \in \mathbf{Z}} (p-r) :b_{m-p-r}c_p c_r: + 2\sqrt{\frac{3-c}{3}} m \sum_{p \in \mathbf{Z}} c_{m-p} I_p + \frac{c}{3}(m^2 - m)c_m, \quad (3.16)$$

and the chiral primary states can be written as  $|\Phi\rangle = |\Psi\rangle \otimes c_1|0\rangle_{gh}$ , where  $|\Psi\rangle$  is a primary state in the "matter + scalar" sector (for a spin-2  $bc$  system  $c_1|0\rangle_{gh}$  is the *true* ghost vacuum annihilated by all the positive modes  $b_n$  and  $c_n$ ).

In both realizations of the  $N = 2$  twisted topological algebra, the relation between the matter central charge and the *topological central charge* turns out to be [7] [8]

$$d = \frac{(c+1)(c+6)}{c-3} \quad (3.17)$$

that implies  $d \leq 1$  or  $d \geq 25$  and  $c \neq 3$ .

Thus we see that the non-critical bosonic string, for  $d \leq 1$  or  $d \geq 25$ , provides a realization of the  $N = 2$  twisted topological algebra.

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